DOING Math with a CAS
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I got started using a CAS with several classes of Algebra 1 and Geometry students when the TI-92 first came out. That certainly doesn't make me the first to use a CAS. But since retiring from teaching high school I find myself doing more and more math with a CAS. I have a math problem; I pick up my CAS (either the V 200 or TI-interactive or Derive – they all do plenty, all you need for high school anyway).

But I notice that I do things differently with a CAS than I do without. That's what I want to discuss with you today: How math is done differently with a CAS than by hand.

Let's start with an example. This first example came to me as a calculus problem, but I later found that the calculus is not needed to solve it. Someone wrote that he spend his spare time (while waiting for the dentist and such) trying to prove that a cubic polynomial is symmetric to its point of inflection. The point of inflection is where the calculus came in, but we can lose the calculus by rephrasing the question:

Example 1: Show that any cubic polynomial has a point of rotational symmetry.

SLIDES 2, 3 and CAS Demo
Extensions:

1. Translate to origin, and show it's an odd function.
2. The point of symmetry is the point of inflection of the graph without calculus.

What's different here? What's the same?

(A few years ago I coined the word "algemetic" to mean all the symbolic manipulation, such as solving equations and simplifying the results, that is a necessary part of doing any algebra problem, but isn't really part of the mathematics involved [1]. That's what we've skipped here by using a CAS. I will use the term here. The algemetic is all the stuff between the input and output on the second line.)

- The *algemetic* is long and complicated – not necessarily difficult – but tedious for sure.
- The mathematical ideas – the idea of symmetry and how it's expressed in an equation – is the same and not likely to change.
- This problem is too difficult to assign to an Algebra 2 class and expect most to get it right by hand; using a CAS they can do the math as opposed to the algemetic.
- Finally, I think this is a nice problem, especially with the extensions. I'll talk more about good CAS problems later.

Example 2: Adapt to common situations – by doing the problems almost the same way.

Demonstrate entry of distance formula

SLIDE 4

Beginning analytic geometry uses about 5 formulas to accomplish a lot of mathematics.

- Slope
- Midpoint
- Distance
- The equation of a line (point – slope form)
- The angle between to lines given their slope.

Each of these can be automated for your CAS.

The slide shows how to enter these and store them to a function (actually a one-line program) in a CAS for future use. In each case, the points are (a,b) and (c,d)

Example 2a **SLIDES 5 and CAS Demo** [2]

Perpendicular bisector theorem: Investigate the set of all points in a plane equidistant from two given points. Given (6,1) and (–5,7) find

a. Equation of points equidistant from them
b. Equation of the line between them
c. Intersection of a. and b
d. Midpoint of line

Example 2b: **SLIDE 6 and CAS Demo**

Example 3: Graph the locus of points in a plane such that the difference of the distances from (–2, 3) and (4, –1) is 3

Cubic symmetry revisited -- Use the <u>midpoint formula</u>

What's different here? What's the same?

- The problems are all standard analytic geometry problems
- As with the cubic, the key was to write the correct equation or equations – just as one would do on paper – only now we can let the CAS solve them.
- The answers can be quite complicated, but that doesn't bother a CAS – even if graphing.

The analytic geometry examples really did things with a CAS the same way as you would do them without a CAS – use the distance formula, write the equation of a line, find the intersection of two lines. Same math, different tools.

The next example uses the new tools to bypass the old techniques.

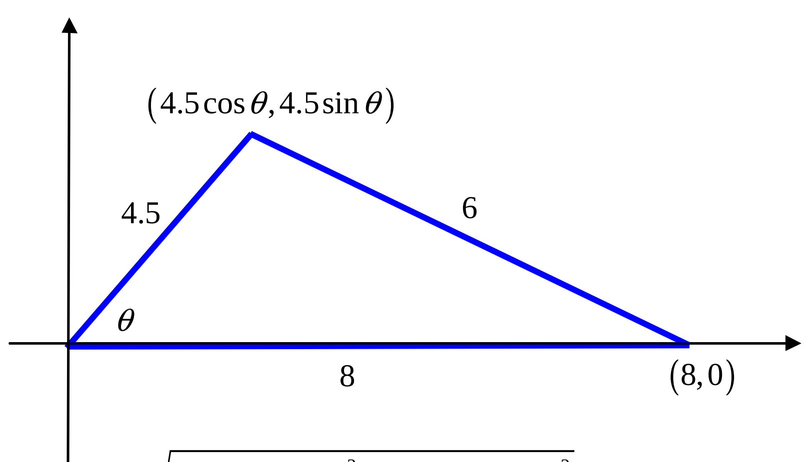
Example 3: Adapt to common situations – by doing the problems new and different ways.

Example 3a Trigonometry, Solve the triangles:

SLIDES 7, 8 and 9 and CAS Demo [3]

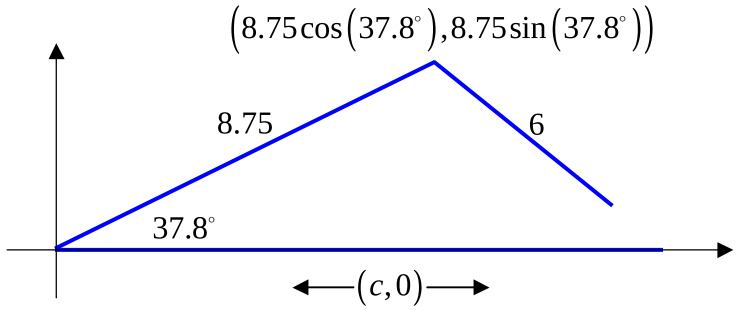
Example 3a (Include if there is time)

SSS: Sides are 4.5, 6 and 8. Find angle opposite the side of 6



$$\sqrt{(4.5\cos\theta - 8)^2 + (4.5\sin\theta - 0)^2} = 6$$

Example 3b: SSA: Angle 37.8 degrees, Side 8.75, next side 6 (or 2 or 10)



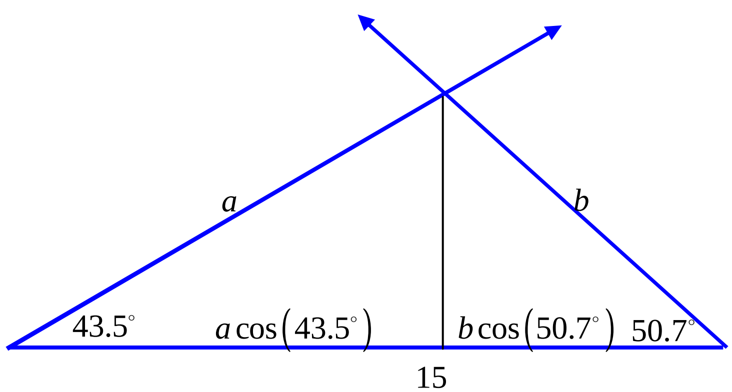
$$\sqrt{\left(8.75\cos\left(37.8^\circ\right)-c\right)^2+\left(8.75\sin\left(37.8^\circ\right)-0\right)^2}=6$$

Or let the distance be 2 or 10

Notice how quickly one can demonstrate the "ambiguous cases."

Example 3c:
ASA: Angle 43.5 degrees, side 15; angle 50.7 degrees

Slide 10



$$a\cos\left(43.5^\circ\right)+b\cos\left(50.7^\circ\right)=15$$

$$a\sin\left(43.5^\circ\right)=b\sin\left(50.7^\circ\right)$$

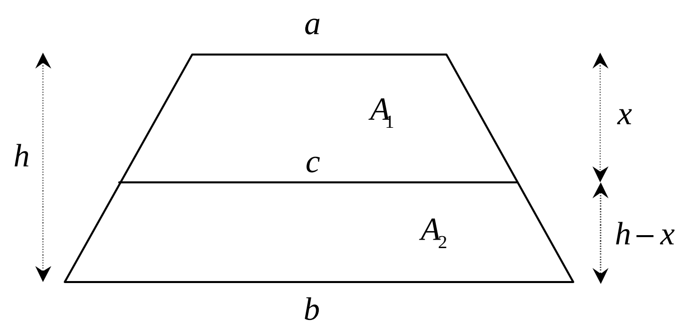
What's different here? What's the same?

- The Laws of Sines and Cosines were not used – the "old" way is actually more complicated to *learn, use* and *teach*.
- The approach is easier to learn, teach and use than the Laws of Sines and Cosines

- The distance formula and Right triangle trigonometry was all that was used here – extended to the coordinate system.
- The algemetic is more complicated with this approach, but as before the CAS can handle the algemetic. (Solving a quadratic even with the quadratic formula is not easy here; that's why the Laws of Sines and Cosines was invented.)
- The "ambiguous case" become obvious (or is the biguous)

Example 4: A new problem done with a CAS.

SLIDES 10 and 11 and CAS Demo [4]



A third example: The Trapezoid Problem.

A trapezoid with base 1 = *a*, and base 2 = *b*. Draw a segment that is parallel to the bases and divides the trapezoid's area A into A1 and A2. Represent the length of the segment in terms of *a* and *b* if A1 = A2. [2] What's different here? What's the same?

- This is an example of "Go for the equations."
- There are other approaches to this problem involving proportions that may be considered more "elegant."
- More "elegant" because here we just plowed through and did the obvious thing, but what's wrong with that?
- The solutions to the equation had to be interpreted and understood; not just accepted.
- Some additional simplifying at the end is helpful, so we can't just blindly depend on the CAS.

Example 5: **SLIDES 12 and 13, Geometry Expressions Demo, CAS Demo** [5]

Altitudes in a right triangle

This is a problem I first saw in playing with *Geometry Expressions* a new piece of software that allows you to do complicated algebra in geometric situations.

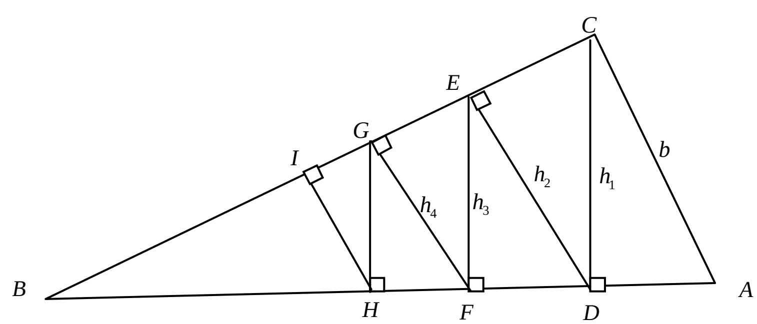
DEMO Geometry Expressions

I am going to do this problem the "hard way" to illustrate a CAS technique that is useful, but not at all helpful when working by hand. I'll show you an easier way at the later, but I did it the hard way first.

You are probably familiar with the theorems concerning the altitude to the hypotenuse of a right triangle. This is related to them.

I want to investigate the successive perpendiculars to the hypotenuses of the smaller triangles that are formed when a new perpendicular is drawn.

SLIDE 13 and CAS Demo



What's different here? What's the same?

- In this problem we substituted successively more complicated expressions into a formula. By hand that doesn't work well at all. By CAS it's not a problem.
- We can just plow through
- Making the algemetic harder is a natural way to do this – if you have a CAS

- While this approach makes the CAS look good it is certainly *not* the easy way to do this problem. Insight still helps. **Show the easy way.**

SLIDE 14 How is DOING Math different with a CAS?

- CAS removes the necessity to do algemetic, so we can concentrate on the mathematics. (All examples)
- Knowing how to use the CAS allows you to improve the CAS by adding routines you need to do standard problems the "usual" way. (Analytic geometry and Trig examples)
- New approaches for doing problems appear once you stop worrying about the algemetic. (Triangle solving)
- 'Go for the equation' – since the CAS can solve (in closed form or not) almost any equation, you needn't worry about how difficult it is to solve. (All except altitudes)
- A CAS will simplify just about any expression so we don't have to avoid complicated expressions – in fact we can complicate what we have, if that makes the flow of work easier. (Altitudes)
- One still needs to know the mathematics to do the set-up and to interpret the answers. (All examples)

SLIDE 15 Implications for teaching **Comments on these points below:**

Good CAS use is a new skill, not just a new tool that students must be taught and encouraged to learn.

To do this we will need

- A willingness to accept new ways of doing problems
- A willingness to accept showing a different kind of work
- A change in the meaning of "simplify."
- A good source of (better) problems for students to attempt.

Comments on points above:

A willingness to accept new ways of doing problems:

- Not just the "new" CAS, but a different style.
- Just as computations with logarithms have left the curriculum other topics may go as well. E.g.: Quadratic formula, Law of Sines and Cosines
- Other ways will be used much more. E.g.: the distance formula

A willingness to accept showing a different kind of work

- Why do we want kids to "show work"? Answer: So that they and we can find the algebra (algemetic) mistakes.
- "Show your work" or rather "show all you computations" should become a thing of the past
- Showing the flow of ideas, what you are thinking, and the justifications for the steps should be required. This is a *writing and communication* skill. Rule of 4
- The "work" may include a print out from the CAS
- And this is good.

- A change in the meaning of "simplify."
  - "Simplify your answer" or just "Simplify" problems are a gray area.
  - Students need to know that things simplify: The CAS will simplify, but students should understand why, say, $\sqrt{50}$ is returned as $5\sqrt{2}$,
  - Students will need practice to with simplifying.
  - Students need to understand why and how things simplify
  - Students need to know that the CAS simplified expression may not be the best simplified form (Trapezoid problem)
  - But they don't need to be good at simplifying.

A good source of (better) problems for students to attempt

- Many of the same problems will remain, but how they are done will change.
- Because they will be able to do more difficult problems, students will need to be good at reading, interpreting, and setting–up problems
- “Story” problems are not going away, but we maybe can have better “stories”
- This seems to me to be one of the big things that needs work – coming up with good new problems and styles of problems.

Thank you for coming and forgive me for preaching to the choir, but then you are the choir. I think the biggest thing we can do at this time is to round up a lot more converts.

Q&A

References

[1]] *Algemetic The Mathematics Teacher* , February 2001, p. 83 – 84. or http://www.linmcmullin.net/PDF_Files/Algemetic.pdf

[2] Analytic Geometry examples – see below.

[3] Triangle Solving examples – see below.

[4] Trapezoid – see below

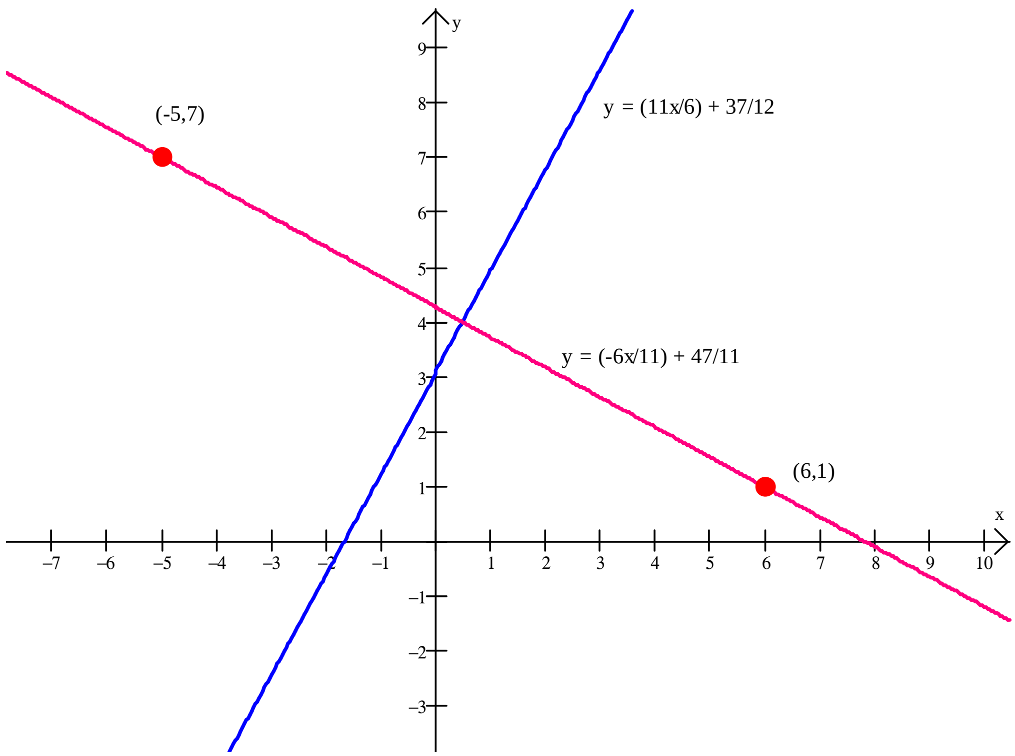
[5] Altitudes in a right triangle – see below.

Most of these examples as well as other are at my web site www.LinMcMullin.net . Click on “Resources” and then “CAS.”

Example 2a: Analytic Geometry

Perpendicular bisector theorem: Investigate the set of all points in a plane equidistant from two given points. Given (6,1) and (–5,7) find

e. Equation of points equidistant from them
f. Equation of the line between them
g. Intersection of a. and b
h. Midpoint of line



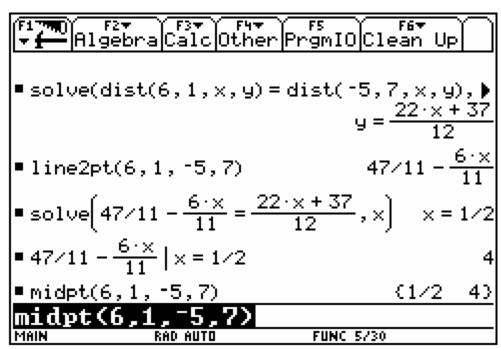
Answers:

a. $y = \frac{11}{6}x + \frac{37}{12}$

b. $y = -\frac{6}{11}x + \frac{47}{11}$

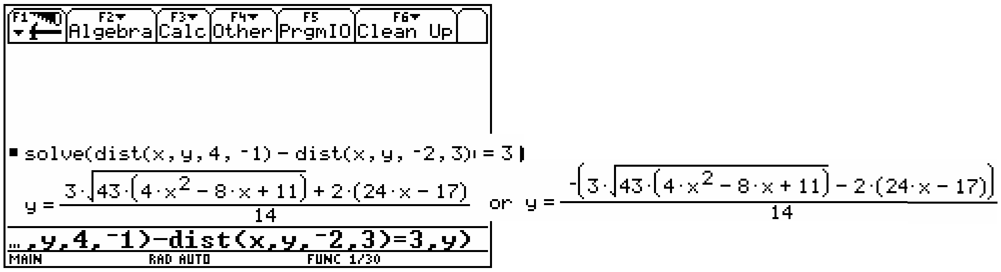
c. $\left(\frac{1}{2}, 4\right)$

d. $\left(\frac{1}{2}, 4\right)$

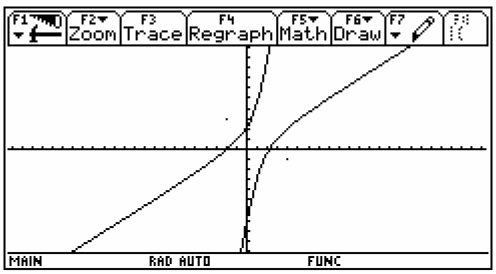


Example 2b: Conic Sections

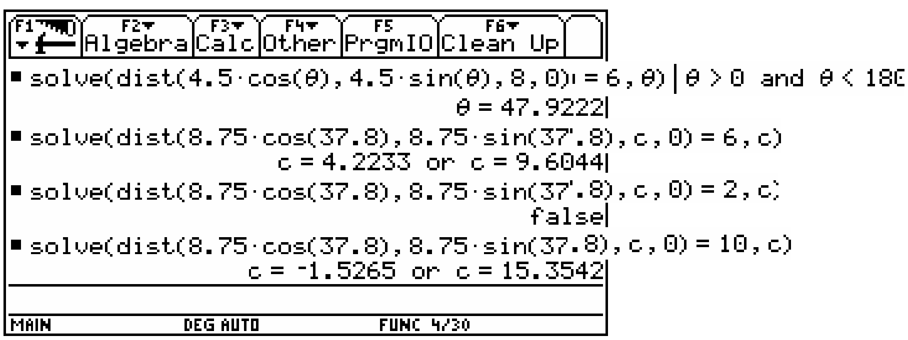
What is the equation of the locus of points such that the difference of the distance from (4, –1) to (–2, 3) is 3?

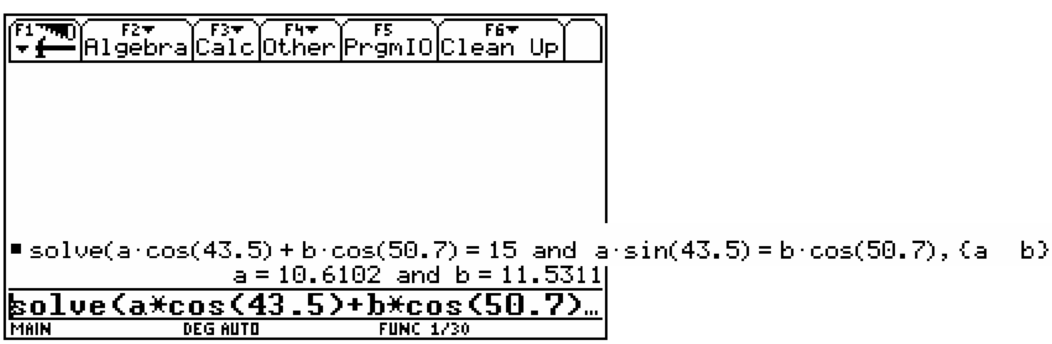


Copy and past both solution to the Equation editor and graph.



Example 3 Triangle solving





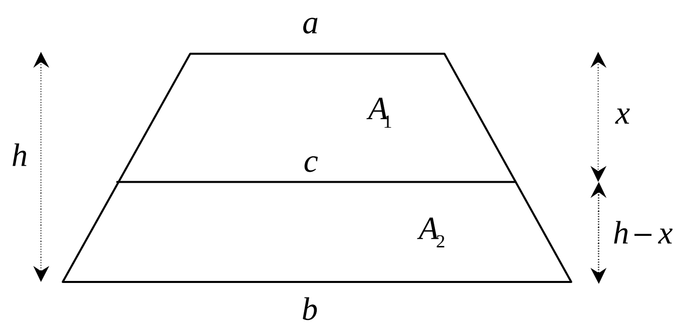
Example 4: The Trapezoid Problem.

A trapezoid with base 1 = a, and base 2 = b. Draw a segment that is parallel to the bases and divides the trapezoid's area A into A1 and A2. Represent the length of the segment in terms of a and b if A1 = A2.

(Suggested by Dr. Jing, St. Anthony High School, Long Beach, CA)

My solution:

Let $0 < a < c < b$, A = area of original trapezoid, and let h = height of the original trapezoid (bases a and b) and let x = height of trapezoid with bases a and c



Then equating the areas gives:

$$A_1 = \tfrac{1}{2}A$$
$$A_1 = A_2$$
or
$$\tfrac{1}{2}(a+c)x = \tfrac{1}{2}\tfrac{1}{2}(a+b)h$$
and
$$\tfrac{1}{2}(a+c)x = \tfrac{1}{2}(c+b)(h-x)$$

This is not an easy system of equation to solve by hand. The only reason you can solve it is that *h* can be anything (it will drop out of the computation) but that is not obvious. So *c* and *x* are the only real "unknowns."[1]

Solving this system for *c* and *x* with a CAS gives

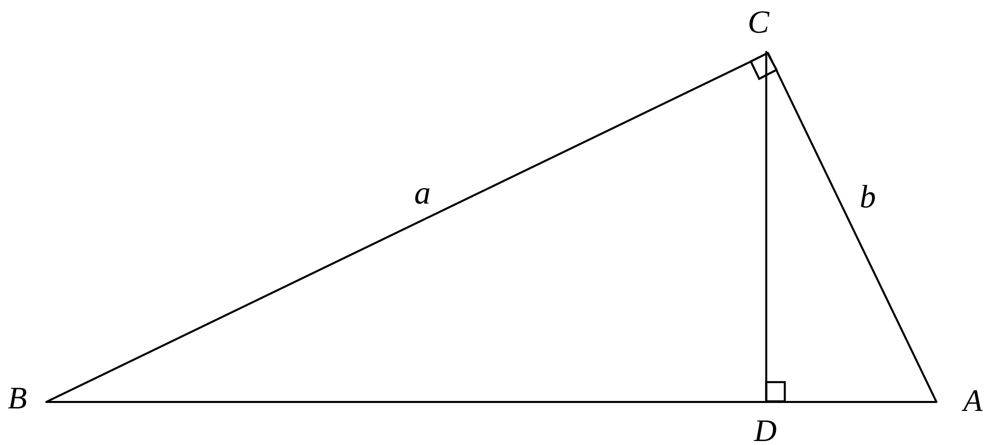
$$c = \frac{\sqrt{2\left(a^2+b^2\right)}}{2} = \sqrt{\frac{a^2+b^2}{2}}$$
$$x = \frac{2a-\sqrt{2\left(a^2+b^2\right)}}{2(a-b)}h = \left(\frac{c-a}{b-a}\right)h$$

[1] Assume you know the height of the original trapezoid. It may be any real number. Since *h* is a factor in all the area formulas it cancels out, but we need to pretend we know it.

Example 5: Altitudes in a Right Triangle

We are given ΔABC a right triangle with sides of *a* and *b* and a right angle at *C*. $\overline{CD}$ is drawn perpendicular to the hypotenuse of ΔABC, so that $\overline{CD} \perp \overline{AB}$.

1. Express *CD* in terms of *a* and *b*



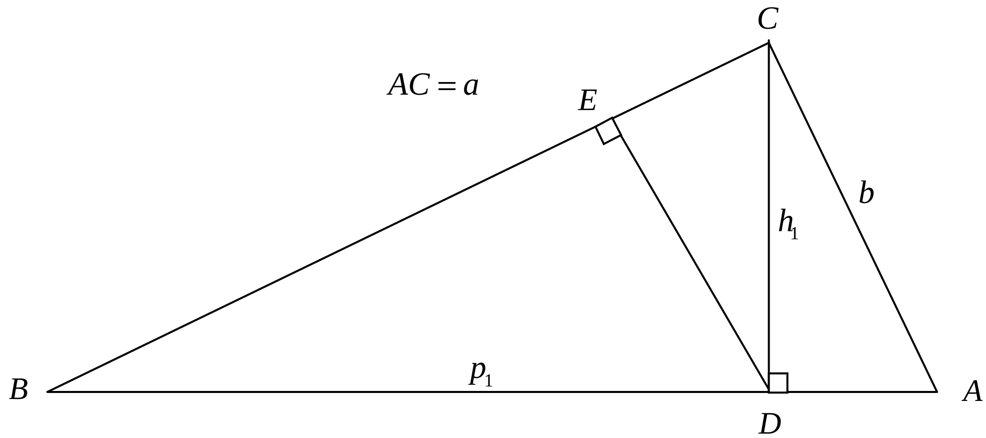
$$AB = \sqrt{a^2 + b^2}$$

Since $\Delta ABC \sim \Delta CBD$

$$\frac{CD}{BC} = \frac{AC}{AB}$$
$$\frac{CD}{a} = \frac{b}{\sqrt{a^2 + b^2}}$$
$$CD = \frac{ab}{\sqrt{a^2 + b^2}} = h_1$$

This is a formula for finding the altitude to the hypotenuse of any right triangle if you know the legs, *a* and *b*. I will store this number to h_1

I want to investigate the perpendiculars to the hypotenuses of the smaller triangles that are formed when a new perpendicular is drawn.

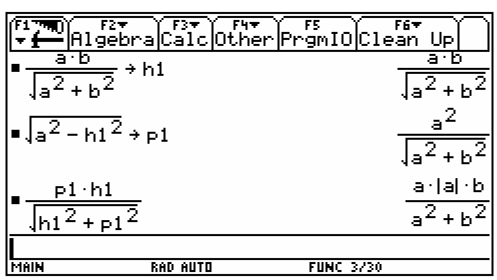


2. Now draw $\overline{DE} \perp \overline{AC}$ as shown above. Find the length of $\overline{DE}$ in terms of *a* and *b*. We can do this using the formula we just found, but first we need to find the length $BD = p_1$. This is a simple Pythagorean Theorem computation. I'll store the answer to p_1.

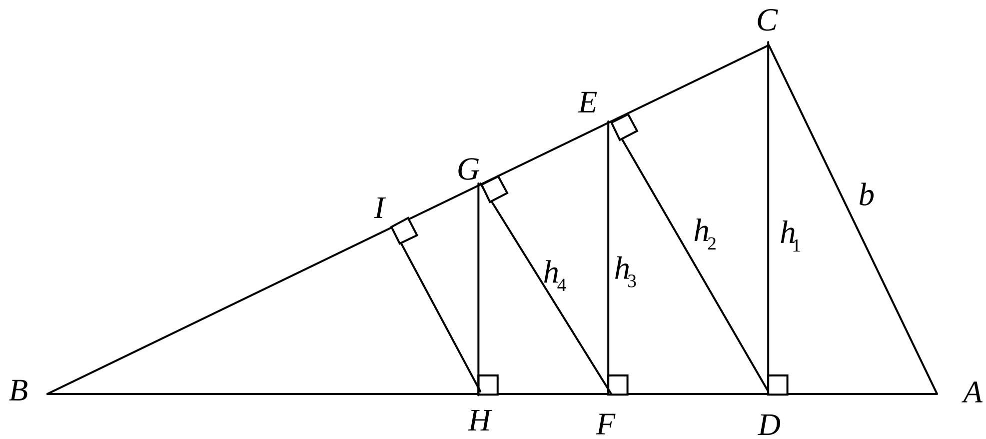
$$\sqrt{a^2 - h_1^2} = p_1$$

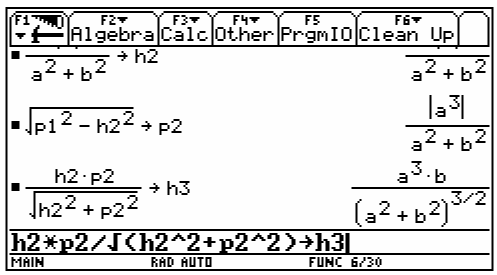
Finally, I'll use the formula found in step 1, to find $ED = h_2$, the altitude to the hypotenuse of right ΔBCD with legs of h_1 and p_1. and the CAS does the rest.

$$h_2 = \frac{p_1 h_1}{\sqrt{p_1^{\ 2} + h_1^{\ 2}}} = \frac{a^2 b}{\sqrt{a^2 + b^2}}$$



3. Continue this process drawing $\overline{EF} \perp \overline{AB}$, $\overline{FG} \perp \overline{AC}$, $\overline{GH} \perp \overline{AB}$ etc., as shown above. Find the length of each segment *EF*, *FG, GH*, etc. always in terms of *a* and *b*, until a pattern emerges.





The pattern is $h_n = \dfrac{a^n b}{\left(a^2+b^2\right)^{n/2}}$

The easy way:

Note that $\angle B \cong \angle ACD \cong \angle CDE \cong \angle DEF \cong \angle EFG \cong \cdots$

Then:

$$h_1 = b\cos B$$
$$h_2 = b\left(\cos B\right)^2$$
$$h_3 = b\left(\cos B\right)^3$$
$$\cdots$$
$$h_n = b\left(\cos B\right)$$
$$= b\left(\frac{a}{\sqrt{a^2+b^2}}\right)^n = \frac{a^n b}{\left(a^2+b^2\right)^{n/2}}$$

