

# AP Calculus Explorations for Teachers

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## GRAPHING

The AP Calculus Course Description indicates that AB and BC calculus students should understand

- Derivatives presented graphically
- The first and second derivatives as functions.
- The corresponding characteristics of  $f$ ,  $f'$ , and  $f''$ .
- The relationship between the increasing and decreasing behavior of  $f$  and the sign of  $f'$ .
- The relationship between the concavity of  $f$  and the sign of  $f''$ .
- Points of inflection as places where concavity changes.
- The analysis of curves.
- Absolute (or global) and relative (or local) extreme values.
- The definite integral of a rate of change over an interval as the change in the quantity over the interval  $\int_a^b f'(x) dx = f(b) - f(a)$ ; accumulation functions.
- The analytical and graphical analysis of functions defined by definite integrals  $f(x) = f(a) + \int_a^x f'(t) dt$ ; accumulation functions.
- The basic properties of definite integrals.

These and other concepts are often tested by questions that give the derivative (or second derivative) and ask students about the properties of the corresponding function. On the AP exams the information about the derivative may be given as an equation, as it is in many textbooks, a graph or even a table.

The derivative given as a graph is an approach is *not* found in many textbooks and, possibly because of this, students have trouble with this when encountering a question like this. Over the years the mean scores on this question have been low; lower than would be expected for such an easy question.

This exploration is intended to help AP Calculus teachers better understand and present the associated ideas, concepts, and ways of thinking to help their students understand these questions.

The annotated table at the end will help you find other questions from past AP exams of this type.

## 1. Exercise 1: Read 2000 AB 3

### The stem:

- You are given the graph of the derivative of a function. The stem describes the obvious features of the graph including the following. Why might you need to know these features? What do they tell you?
  - Domain
  - Horizontal tangent lines
  - Vertical tangent lines
  
- Features can you read or assume from the graph. Why might these be important? What do they tell you?
  - Roots of the derivative
  - Places where the derivative is positive or negative
  - Places where the derivative increases and decreases
  - Maximum or minimum points *of the derivative*
  - Concavity *of the derivative*
  - Relative size (area) of the regions

Many books suggest setting up a table similar to the one on the next page to analyze the relationship between a function and its first and second derivatives. Other books suggest a number line analysis. While it is intended that this question can be answered directly from the graph, let's look at all the approaches.

On the next page do a table and a number line analysis of 2000 AB 3.



### The questions

- Parts (a) and (b) ask for local minimums and maximums.
  - What do you look for on the graph of the derivative allows you to find
    - Relative maximums?
    - Relative minimums?
  - What can you write for an acceptable justification of a local extreme value?
- Part (c) asks where the second derivative is negative.
  - How can you tell from the graph of the derivative?
  - How can you be sure of the end points of the intervals?
  - What does this tell you about the graph of the function?
- Part (d) Asks for the location of the absolute maximum value
  - What are the *candidates* for an extreme value
  - What features of the graph helps answer this question.
  - What can you write for a justification?
  - Why is the interval in the stem and in this part a closed interval, while in the other parts there is an open interval?

**2. Another plan for 2000 AB 3:**

Consider the accumulation function  $f(x) = \int_{-7}^x f'(t) dt, -7 \leq x \leq 7$ .

Tell me about the value of  $f$  as the vertical line at  $x$  moves right

- $f(-7) =$
- from  $x = -7$  to  $-5$
- from  $-5$  to  $-1$
- from  $-1$  to  $3$
- from  $3$  to  $5$
- from  $5$  to  $7$

How does this analysis help you find extreme values?

Now consider the shape (concavity) of the graph as  $x$  moves right

- from  $-7$  to about  $-3$
- from about  $-3$  to about  $2$
- from about  $2$  to  $3$
- from  $3$  to  $5$
- from  $5$  to  $7$

Discussion of the scoring standard

**3. Winplot activity:** Go to <http://www.linmcmullin.net/Technology.html> and download the “Concavity Demonstration” Winplot file. Open with Winplot and push CTRL+SHIFT+N to view the instructions.

#### **4. Exercise 2: 2006 AB 3**

**The stem:**

- You are given the graph of a function,  $f$ , and a second function,  $g$ , defined as a definite integral of  $f$ . You are told the graph consists of 6 segments and the end points of the segments are given.
  - What is the domain of  $f$
  - What is the range of  $f$
  - Is  $f$  odd, even, or neither?
  - What is the relationship between  $f$  and  $g$ ? State the relationship two ways

On the next page do a table and a number line analysis of 2006 AB 3

**2006 AB 3 Analysis by table (abbreviate and leave cells empty if nothing can be determined – this is important too) for one period:**

$x$	0	(0,1)	1	(1,2)	2	(2,3)	3	(3,4)	4	(4,5)	5
$f(x)$											
$f'(x)$											
$f''(x)$											

**2006 Number line analysis for one period:**



**The parts:**

- Part (a) asks for 3 values. Explain how you find
  - $g(4)$
  - $g'(4)$
  - $g''(4)$
  - Find  $g(0) = \underline{\hspace{2cm}}$ ,  $g'(0) = \underline{\hspace{2cm}}$ ,  $g''(0) = \underline{\hspace{2cm}}$
  - Find  $g(-1) = \underline{\hspace{2cm}}$ ,  $g'(-1) = \underline{\hspace{2cm}}$ ,  $g''(-1) = \underline{\hspace{2cm}}$
- Part (b) asks if there is an extreme value at  $x = 1$ .
  - What kind of extreme value is it?
  - How can you justify the answer?
  - List all other extreme values and classify them as relative or absolute maximums or minimums.
- Part (c) changes the domain of  $f$  and makes the function periodic with a period of 5. It gives the correct value of  $g(5)$  and then asks for information from outside the original domain.
  - Explain how to find  $g(10)$
  - This function is similar to  $y = \sin(x)$ , but for the sine function the value at the end of each period is zero. Why is this not true for  $g$ ?
  - What information do you need to write the equation of a line?
  - What considerations go into finding  $g(108)$ ?

Discussion of the scoring standard

## **5. Matching the antiderivative.**

**6. Exercise 3:** Now you're on your own: Do 2003 AB 5. Timed: 15 minutes.  
Discussion of Scoring Standards

**7. Exercise 4:** Many of the skills needed to answer these questions are taught in the 4-years of mathematics prior to studying the calculus. This exercise will give you some practice identifying these skills and show you how to start preparing your students for AP Calculus long before they start the class.

Return to 2006 AB 3 and write 2 questions based on this questions or its graph that would be suitable to use at each level listed below. You may, if necessary, revise the stem or graph slightly.

**Algebra 1**

**Geometry**

**Algebra 2**

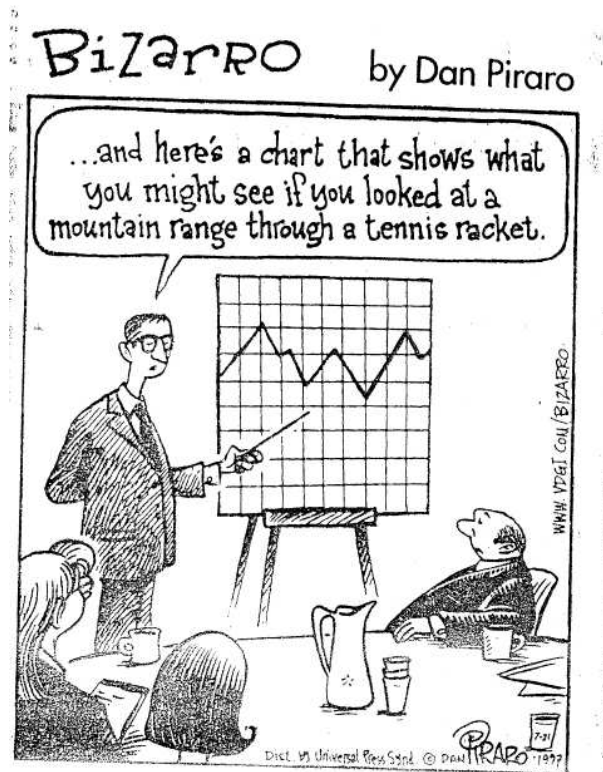
**Trigonometry**

**Precalculus**

## **8. Brainstorm:**

What can you have your STUDENTS doing on a regular basis so that they will master these concepts?

- 1.
- 2.
- 3.
- 4.
- 5.



For similar explorations of other AP Calculus free-response type problems see

[http://www.linmcmullin.net/AP\\_Calculus\\_NEW.html](http://www.linmcmullin.net/AP_Calculus_NEW.html) or go to

[www.LinMcMullin.net](http://www.LinMcMullin.net) > RESOURCES > AP\* CALCULUS

### Index to AP Free-response questions Topic 4.

The questions themselves can be found at AP Central. Click on “The Exams” and then on “Exam Questions” then “Calculus AB” or “Calculus BC” in the table. or go to [http://apcentral.collegeboard.com/apc/members/exam/exam\\_questions/1997.html](http://apcentral.collegeboard.com/apc/members/exam/exam_questions/1997.html)

Topic 4: Information from graph or about the graph, family of functions, $f, f', f''$		
Year & Question	GCA	Comments
1998 AB2- BC2	Yes	Limits, max/min, “family”
1999 AB4	Yes	From equation, 2-DT, function defined by other function
1999 AB5-BC5	Yes	From graph: Max/min, inc/dec, tangent line,
2000 AB3	Yes	From graph: $f, f', f''$ , inc/dec, POI
2001 AB3-BC3	Yes	Particle motion (q.v.) setting
2001 AB4-BC4	No	From derivative equation: Max/min, tangent line, concavity
2002 AB1 BC1 (c)	Yes	From Equation; Includes max/min area/volume
2002 AB4-BC4	No	Given $f'$ graph: inc/dec, concavity, draw graph $f$
2002 BC5 (d)	No	Slope field w/ draw solution, Euler, max/min
2002 AB2 B	Yes	Given equation of $f'$ : Max/min, accumulation
2002 AB4-BC4 B	No	Given graph $f$ : values, inc/dec, concavity, Trap rule.
2003 AB3	Yes	Given table and graph: Diff quotient, Riemann sum, max/min, average value, accumulation
2003 AB4-BC4	No	Given graph $f'$ : inc/dec, tangent line, FTC by area.
2003 AB5 B	No	Given graph $f$ : values, average values, MVT, POI
2004 AB5	No	Given graph $f$ : values, FTC, max/min, POI
2004 AB2 B	Yes	From equation; Accumulation setting, max/min, inc/dec.
2004 AB4-BC4 B	No	Given graph $f'$ : inc/dec, product rule
2005 AB4	No	Tabular information about $f, f', f''$ , FTC, max/min, draw graph, POI
2005 AB4-BC4 B	No	Given graph $f'$ : values, FTC from graph area, inc/dec.
2006 AB3	Yes	Periodic function, accumulation, tangent line
2006 AB 2 B	Yes	Concavity, max/min, tangent line
2006 AB4-BC4 B	Yes	From graph, derivative, area