The Two Cone Problem

"Find the maximum sum of the volumes of two cones that you make out of a construction paper disc with an area of S."

Suggested by Dr. Jing, St. Anthony High School, Long Beach, CA AP Calculus EDG 11/12/06

Solution (by Lin McMullin)

Assumption: The cones are formed by cutting a sector from a circular disk and forming both pieces into cones. Another approach which may involve some waste may be possible.

For the moment assume S = 1

Then the radius of the disk is $r=\sqrt{\frac{1}{\pi}}$ and this becomes the slant height of both cones.

The circumference of the disk is $2\pi\sqrt{\frac{1}{\pi}}=2\sqrt{\pi}$

Let x = the arc length of the sector cut from the disk. $0\le x\le 2\sqrt{\pi}$ or about $0\le x\le 3.5449$

Cone 1:

Slant height = $\sqrt{\frac{1}{\pi}}$

Circumference = x

Radius of base $r_1=\dfrac{x}{2\pi}$

Height = $\sqrt{\left(\sqrt{\dfrac{1}{\pi}}\right)^2-\left(\dfrac{x}{2\pi}\right)^2}=\sqrt{\dfrac{1}{\pi}-\left(\dfrac{x}{2\pi}\right)^2}$

Volume is $V_1=\frac{\pi}{3}\left(\frac{x}{2\pi}\right)^2\sqrt{\frac{1}{\pi}-\left(\frac{x}{2\pi}\right)^2}=\dfrac{x^2}{12\pi}\sqrt{\dfrac{1}{\pi}-\left(\dfrac{x}{2\pi}\right)^2}$

Cone 2:

Slant height = $\sqrt{\frac{1}{\pi}}$

Circumference = $2\sqrt{\pi}-x$

Radius of base $r_2=\dfrac{2\sqrt{\pi}-x}{2\pi}$

Height = $\sqrt{\left(\sqrt{\frac{1}{\pi}}\right)^2-\left(\frac{2\sqrt{\pi}-x}{2\pi}\right)^2}$

Volume is $V_2=\frac{\pi}{3}\left(\frac{2\sqrt{\pi}-x}{2\pi}\right)^2\sqrt{\left(\frac{1}{\pi}\right)-\left(\frac{2\sqrt{\pi}-x}{2\pi}\right)^2}=\dfrac{\left(2\sqrt{\pi}-x\right)^2}{12\pi}\sqrt{\left(\dfrac{1}{\pi}\right)-\left(\dfrac{2\sqrt{\pi}-x}{2\pi}\right)^2}$

The quantity to be maximized is

$$V = V_1 + V_2 = \frac{x^2}{12\pi}\sqrt{\frac{1}{\pi} - \left(\frac{x}{2\pi}\right)^2} + \frac{\left(2\sqrt{\pi} - x\right)^2}{12\pi}\sqrt{\left(\frac{1}{\pi}\right) - \left(\frac{2\sqrt{\pi} - x}{2\pi}\right)^2}$$

Then (computation done with TI-Interactive)

$$\frac{x^2}{12\cdot\pi}\cdot\sqrt{\frac{1}{\pi} - \left(\frac{x}{2\cdot\pi}\right)^2} + \frac{\left(2\sqrt{\pi} - x\right)^2}{12\cdot\pi}\cdot\sqrt{\frac{1}{\pi} - \left(\frac{2\sqrt{\pi} - x}{2\cdot\pi}\right)^2}$$

$$\frac{x^2\cdot\sqrt{4\cdot\pi - x^2}}{24\cdot\pi^2} + \frac{\left(x - 2\cdot\sqrt{\pi}\right)^2\cdot\sqrt{-x\cdot\left(x - 4\cdot\sqrt{\pi}\right)}}{24\cdot\pi^2}$$

$$\frac{d}{dx}\,ans$$

$$\frac{x\cdot\sqrt{4\cdot\pi - x^2}}{12\cdot\pi^2} - \frac{x^3}{24\cdot\pi^2\cdot\sqrt{4\cdot\pi - x^2}} + \frac{\left(x - 2\cdot\sqrt{\pi}\right)\cdot\sqrt{-x\cdot\left(x - 4\cdot\sqrt{\pi}\right)}}{12\cdot\pi^2} - \frac{\left(x - 2\cdot\sqrt{\pi}\right)^3}{24\cdot\pi^2\cdot\sqrt{-x\cdot\left(x - 4\cdot\sqrt{\pi}\right)}}$$

$$\text{solve}\left(ans = 0, x\right)$$

$$x = 2.39631 \text{ or } x = 1.77245 \text{ or } x = 1.1486$$

$$\left(\frac{x^2\cdot\sqrt{4\cdot\pi - x^2}}{24\cdot\pi^2} + \frac{\left(x - 2\cdot\sqrt{\pi}\right)^2\cdot\sqrt{-x\cdot\left(x - 4\cdot\sqrt{\pi}\right)}}{24\cdot\pi^2}\right) \mid x = \left\{2.39631, 1.77245, 1.1486\right\}$$

$$\left\{.082007, .081434, .082007\right\}$$

These are the volumes at the three critical points (end point volumes = 0 i.e. no cones). The first and third are the same because they represent the solution when cone 1 and cone 2 are interchanged by cutting so that $x = 2\sqrt{\pi} - 1.1486 = 2.39631$

Finally, since the area was multiplied by a factor of $\frac{1}{S}$ to make it =1, we must multiply the volumes by a factor of $S^{\frac{3}{2}}$ t o return to the original problem size.

The maximum volume is $(0.082007)S^{\frac{3}{2}}$

Dr. Jing's solution:

In a message dated 11/14/2006 8:13:46 P.M. Eastern Standard Time, sahsjing@yahoo.com writes:
Thank you all for sharing your ideas to get the problem done.

I wrote the problem with two purposes: 1) to write a single valuable function to represent the volume of two cones made out of a unit disc, to find the maximum and to compare the difference between computing by hand and computing by a graphing calculator; 2) to magnify or reduce from a unit disc to a disc with an area S.

Let R1 and R2 be the radius of the two cones made out of a disc with an area S. Then we have R1+R2 = r, where r is the radius of the disc.

V = f(r, R1, R2) = (1/3)(pi)[R1^2 sqrt(r^2-R1^2) + R2^2 sqrt(r^2-R2^2)]

For unit disc,
r = 1.=> R2 = 1-R1.
V = g(R1) = (1/3)(pi)[(R1^2) sqrt(1-R1^2) + (1-R1)^2 sqrt(1-(1-R1)^2)]
With a graphing calculator (TI-83 plus), I got V = Vmax at

R1 = 0.3240, or 1-0.3240.

Therefore, Vmax = g(0.3240).

For the disc with an area S,
V*max = f(r , 0.3240r, (1-0.3240)r) = f(sqrt[S/pi] , 0.3240 sqrt[S/pi], (1-0.3240) sqrt[S/pi]) = h(S).

Therefore, V*max is the function of S only.

And

In a message dated 11/15/2006 6:30:34 P.M. Eastern Standard Time, sahsjing@yahoo.com writes:

Thank you for sharing your opinions with me.

First, I would like to tell you that I read your solution carefully and found that it leads to the same results as my approach would do. The only difference between the two approaches is that you used S = 1, while I used r = 1. The reason I used r = 1 is that I found this way it is easier to simplify the total volume expression, and that the expression is more readable.

Second, I want to say that I like your final formula for the maximum volumes. It is neat! I should have thought about the way of using similarity.

Third, I want to make it clear that the cut I made is exactly the same as yours. Therefore, there is no waste.

Now let me show why your approach is the same as my approach.

The circumference of the disc is 2 pi r
The circumferences of the two cones are 2 pi R1 and 2 pi R2.

Since the two sectors are cut from the center of the disc, we have 2 pi r = 2 pi R1 + 2 pi R2, which leads to r = R1 + R2.

Therefore, R1 and R2 are exactly the radii of the two cones.

Be the way, the slant height of the two cones in my approach is the same as the radius of the disc, as shown in V = f(r, R1, R2) = (1/3)(pi)[R1^2 sqrt(r^2-R1^2) + R2^2 sqrt(r^2-R2^2)]. [H1 = sqrt(r^2-R1^2) , and H2 = sqrt(r^2-R2^2)]

Solution by Ken Sterling using the central angle:

In a message dated 11/13/2006 8:21:50 P.M. Eastern Standard Time, ksterling@mindspring.com writes:
Wow. My first reaction to the problem was that it would me a messy confirmation that the max would occur using half the circle for each cone.

I wrote the sum of the volumes as a function the central angle (t) of a circular sector cut for one of the cones.

V(t) = Const * [t^2 * sqrt(4pi^2 - t^2) + (2pi - t)* sqrt(4pi*t - t^2)
on 0 <= t <= 2pi.

V(t) is 0 at 0 and 2pi as expected, but there is where my expectations for the graph ended.

Using technology, the graph appears to have three horizontal tangents. One is at pi but that looks to be a relative min. Two max's occur. One at approx. 2.03584 and the other at 4.24753. The graph of the derivative (again using TI-89) confirms three horizontal tangent lines at those locations.

I did not try to solve the derivative set to zero by hand, but I found the symbolic derivative with the TI89 and asked it to solve deriv = zero. It just went busy for about ten minutes and never produced an answer.

The apparent answer is so unintuitive that I have to wonder about mistakes in my methods of something else going on.

Wow again.

Nice problem.

Solution by Mark Snyder:

In a message dated 11/15/2006 7:56:06 P.M. Eastern Standard Time, msnyder@fsc.edu writes:

The only thing I would add to this discussion is that the function to be minimized can be made to look nicer by taking (1/2)-x to be the fraction of the original disk cut out to make cone #1, so (1/2)+x is the fraction cut out to make cone #2. This means that the sum of the volumes of the cones must be symmetrical under x---> - x, i.e., it must be an even function of x.

Then when the cones are formed, the volume to be minimized is proportional to

F = (2x - 1)^2 sqrt[4 - (2x - 1)^2] + (2x + 1)^2 sqrt[4 - (2x + 1)^2].

This makes the function to be minimized look nicer, and the even-itude obvious. It also leads to an exact solution.

From a graph of the function, one sees that it has three local extrema. Owing to the above symmetry, this means that one of the extrema must be at $x = 0$, while the other two are at opposite values of x. The extremum at $x = 0$ is a local minimum, corresponding to cutting the disk in half (the solution I thought I would find as the maximum...), while the other two are maxima, and occur at approximately $x = (\pm)0.176$, i.e., make the cut at an angle of about 118.6 degrees. That works out to a maximum volume of about $0.08 \cdot S^{3/2}$, as others have found.

In fact, the problem can be solved analytically: with some manipulations, if we set $F' = 0$ and square to remove the square roots, we find that $y = 4x^2$ obeys the cubic equation

$$9y^3 - 87y^2 + 51y - 5 = 0.$$

As a cubic equation, this has an analytic, exact solution (which I will *not* quote...).

While this is a cute problem, I was hoping that there was some "aha" method of finding the solution, not a graphical/numerical/analytical one. I looked for such a solution, but couldn't find one.